SEISMIC BEARING CAPACITY FACTORS BY ZERO EXTENSION LINE METHOD

A. GHARHAMANI1, J.B. BERRILL2
1Visiting Professor from Shiraz University, Iran
2University of Canterbury, Christchurch, New Zealand

SUMMARY

The seismic bearing capacity of soils is evaluated by the method of zero extension line. The inclination of footing reaction as well as surcharge due to horizontal acceleration is considered. The equilibrium equation along the zero extension line is used to evaluate $N_u$, $N_s$, and $N_r$, the cohesion, surcharge and gravity bearing capacity factors. It is found that due to seismic acceleration, $N_u$, $N_s$, and $N_r$ are reduced and bearing capacity is further reduced due to inertia of the soil mass.

INTRODUCTION

The seismic bearing capacity of foundations have been studied recently by Sarma and Lassifellis (1990) and Richard, Ellis and Bodhe (1993). The method of zero extension line (lines with linear strain equal to zero), first proposed by Roscoe (1970) and later used by James and Bumsby (1971) for predicting the strain pattern behind retaining walls was extended by Hadiabahi and Ghabramani (1970) for static earth pressure. The extension to dynamic earth pressure was carried out by Ghabramani and Clemence (1990). Bearing capacity was dealt with by the zero extension line method by Babusco and Ghabramani (1987) and (1994). Ansar and Ghabramani (1993) treated active dynamic pressure on retaining walls. The method was applied to seismic bearing capacity of clay soils by Ghabramani and Berrill (1993). It is the purpose of the present paper to study the seismic bearing capacity of general soils having both cohesion and friction. Both the surcharge q and the footing pressure p have horizontal components during seismic loading. The tangent of inclination angle equals $k$, which is the horizontal fraction of gravity acceleration due to seismic loading. The effect of horizontal inertia component has also been included in the analysis.

THEORY

The theoretical development of the method of zero extension line has been dealt with in the references cited. It is shown that in limiting equilibrium there are two zero extension line (ZEL) directions which make angle $\xi = \frac{\alpha - \beta}{2}$ with the direction of $\alpha$ axis of major compressive stress (Fig. 1). Angle $\alpha$ is the angle of dilation of soil.

The two directions are given by the following equations:

$$ \frac{dz}{dx} = \tan (\beta - \xi) $$  \hspace{1cm} (1)

$$ \frac{dz}{dx} = \tan (\beta + \xi) $$  \hspace{1cm} (2)

where $\beta$ is the angle between the direction of $\alpha$ and the $x$ axis.
The equilibrium equations along the two zero extension line directions are

On the plus line

\[ du + 2 \left( u \tan \phi + c \right) \left[ a \, dt + \eta \, \frac{d\phi}{d\xi} \, d\xi \right] = \]
\[ - X \dot{\beta} \left[ \tan \phi \, dx + \dot{\alpha} \, dx \right] - Z \dot{\beta} \left[ \tan \phi \, dx + \dot{\alpha} \, dz \right] \]
\[ + \left( u - c \tan \phi \right) \left[ \tan \phi \, d\phi - \frac{1}{\cos \phi} \frac{d\phi}{d\xi} \, d\xi \right] \]
\[ + \left[ \tan \phi \, dc - \frac{1}{\cos \phi} \frac{dc}{d\xi} \, d\xi \right] \]  

(3)

On the minus line

\[ du - 2 \left( u \tan \phi - c \right) \left[ a \, dt + \eta \, \frac{d\phi}{d\xi} \, d\xi \right] = \]
\[ X \dot{\beta} \left[ \tan \phi \, dx + \dot{\alpha} \, dx \right] - Z \dot{\beta} \left[ \tan \phi \, dx + \dot{\alpha} \, dz \right] \]
\[ + \left( u - c \tan \phi \right) \left[ \tan \phi \, d\phi - \frac{1}{\cos \phi} \frac{d\phi}{d\xi} \, d\xi \right] \]
\[ + \left[ \tan \phi \, dc - \frac{1}{\cos \phi} \frac{dc}{d\xi} \, d\xi \right] \]  

(4)

where

\[ \dot{\alpha} = \frac{1 - \sin \phi \cos \psi}{\cos \phi \cos \psi} \]
\[ \dot{\eta} = \frac{\sin \phi - \sin \psi}{\cos \phi \cos \psi} \]
\[ \dot{\beta} = \frac{\cos \phi}{\cos \phi} \]
\[ \lambda = \frac{\sin \phi - \sin \psi}{\cos \phi} \]
\[ \mu = \frac{\sigma}{2} \]

and X and Z are the horizontal and vertical body forces.
It should be noted that for any function $f$, along the plus direction
\begin{equation}
\tan \phi \, df - \frac{1}{\cos \phi} \frac{df}{dc} \, dc = \lambda \, df + \beta \left( \frac{df}{dx} \, dx - \frac{df}{dz} \, dz \right)
\end{equation}
\(6\)
on a minus line
\begin{equation}
\tan \phi \, df - \frac{1}{\cos \phi} \frac{df}{dc} \, dc = \lambda \, df + \beta \left( \frac{df}{dx} \, dx - \frac{df}{dz} \, dz \right)
\end{equation}
\(7\)
During earthquake loading, the surcharge $q$ and footing pressure $p$ are with inclination $i$ (Fig. 2) such that $\tan i = k_{eq}$, which is the horizontal fraction of gravity acting due to the seismic loading.

![Figure 3. Stress characteristics and ZEL network near singularity](image-url)

The zero extension line (ZEL) field is composed of three fields: a Rankine zone AOB, a radial Coulomb zone C, and Coulomb zone OCB under the footing with width equal to $h$. It should be noted that the zero extension line network is straight in Rankine and Coulomb zones and in Coulomb zone it is composed of straight radial lines and spirals with angle $v$ for minus direction.

The equations of equilibrium should be integrated along ABCD to get bearing capacity. The result is presented in the classical form as
\begin{equation}
p = c \, N_e + q \, N_x - \frac{1}{2} \gamma b N_e
\end{equation}
\(8\)
For a constant $c$, soil it is simpler to define a reduced mean stress $u = n + c \cot \phi$. Then the equilibrium equation along the ZEL in the $z$ direction becomes:
\begin{equation}
dz - 2 \tan \phi \, u \left( \dot{\theta} \, dz + \eta \, \frac{d \phi}{dc} \, dz \right) =
X \dot{b} \left( \tan \phi \, dz + \dot{a} \, dz \right) - Z \dot{b} \left( \tan \phi \, dz + \dot{a} \, dz \right)
\end{equation}
\(9\)
In this equation $Z = n$, and $X = k_{eq}$ for the seismic case, where $h$ is unit weight.

In the following analysis as customary for bearing capacity calculations, $N_e$ and $N_x$ are evaluated first using a soil with $\gamma = 0$, then $N_e$ is evaluated using a soil with $c = 0$.

**EVALUATION OF $N_e$ AND $N_x$**

For the soil with $\gamma = 0$ the equation of equilibrium reduces to
\begin{equation}
dz - 2 \tan \phi \, u \left( \dot{\theta} \, dz + \eta \, \frac{d \phi}{dc} \, dz \right) = 0
\end{equation}
\(10\)
Now \( \frac{\partial \delta}{\partial C} \) is non-zero through the singularity at point \( O \).

Throughout the singularity, the stress characteristics (Fig. 3) are composed of straight radial lines and logarithmic spirals with spiral angles \( \phi \) for minus stress characteristics.

The Mohr circle for surcharge \( q \) is shown in Figure 4 and the circle for bearing pressure \( p \) is shown in Figure 5.

![Figure 4. Surcharge Mohr circle](image)
![Figure 5. Bearing pressure Mohr circle](image)

However, the ZEL is composed of two sets of logarithmic spirals: for the minus direction, the spiral angle is \((\phi + \psi)/2\) and for the plus direction, the spiral angle is \((\phi - \psi)/2\). Thus these equations are valid:

\[
\begin{align*}
(\cdot) \text{ stress characteristic:} & \quad r = r_0 e^{\omega \theta} \\
(\cdot) \text{ ZEL:} & \quad r = r_0 e^{\omega (\pi/2 - \theta)} \\
(\cdot) \text{ ZEL:} & \quad r = r_0 e^{\omega (\pi/2 + \theta)}
\end{align*}
\]

where \( r_0 \) is the original radial distance measured from \( O \); \( r \) is the radial distance to a point on the spiral making an angle \( \theta \) with \( r_0 \). Then we get:

\[
\begin{align*}
(\cdot) \text{ ZEL:} & \quad dr = r_0 e^{\omega (\pi/2 - \theta)} \tan \theta \cot \phi \, d\theta = r \tan \phi \, d\theta \\
(\cdot) \text{ ZEL:} & \quad dr = -r_0 e^{\omega (\pi/2 + \theta)} \cot \theta \tan \phi \, d\theta = -r \cot \phi \, d\theta \\
(\cdot) \text{ ZEL:} & \quad dr = r_0 e^{\omega (\pi/2 - \theta)} \tan \frac{n}{2} \, d\theta = -r \tan \left( \frac{\phi + \psi}{2} \right) \, d\theta
\end{align*}
\]

Now, since

\[
\begin{align*}
\frac{\partial \delta}{\partial C} &= \varepsilon \tan \left( \frac{\phi + \psi}{2} \right) \quad \text{and} \quad \frac{\partial \sigma}{\partial C} = \varepsilon \tan \left( \frac{\phi - \psi}{2} \right)
\end{align*}
\]

then

\[
\begin{align*}
\frac{\partial}{\partial C} \frac{\partial \sigma}{\partial C} &= \frac{1}{r} \cos \left( \frac{\phi + \psi}{2} \right) \quad \text{and} \quad \frac{\partial}{\partial C} \frac{\partial \varepsilon}{\partial C} = \frac{1}{r} \cos \left( \frac{\phi - \psi}{2} \right)
\end{align*}
\]
Therefore:

\[
\frac{d}{\delta} \frac{\partial \theta}{\partial \xi} + \eta \frac{\partial \theta}{\partial \zeta} - \frac{\partial \theta}{\partial \zeta} + \left[ \frac{\partial}{\partial \xi} \left( \frac{\eta}{\xi} \right) \right] = \frac{\partial \theta}{\partial \zeta} = \frac{\sin(\phi + \frac{\phi}{2})}{\cos(\phi + \frac{\phi}{2})} \nonumber
\]

Algebraic manipulation shows that the quantity in the first term is equal to unity. Thus in the vicinity of the singularity

\[\frac{d}{\theta} \frac{\partial \theta}{\partial \xi} + \eta \frac{\partial \theta}{\partial \zeta} - \frac{\partial \theta}{\partial \zeta} = 0 \tag{15}\]

and the equation of equilibrium becomes

\[d\phi + 2 \tan \phi \, d\phi = 0 \nonumber\]

This is the famous equation of Sokolovski for change of \( \phi \) along a singularity.

From the figures it is clear that theta for surcharge

\[\theta_q = \frac{\delta}{2} \quad \text{and} \quad \theta_q = \frac{\pi}{2} - \frac{\beta}{2} \tag{17}\]

\[\alpha_1 = \frac{\pi}{2} - \frac{\nu}{2} - \frac{\delta}{2} \quad \text{and} \quad \alpha_1 = \frac{\pi}{2} - \frac{\nu}{2} - \frac{\beta}{2} \tag{18}\]

\[\tan \theta_q = \tan \left( \frac{\alpha_1}{q + c \cot \phi} \right) \quad \text{and} \quad \tan \theta_q = \tan \left( \frac{\alpha_1}{p + c \cot \phi} \right) \nonumber\]

If we call \( c/p = e_m \) and \( c/q = e_m \), then

\[i_\theta = \tan^{-1} \left[ \frac{1}{1 + \left( e_m \tan \phi \right) i_\theta} \right] \quad \text{and} \quad i_\theta = \tan^{-1} \left[ \frac{1}{1 + \left( e_m \tan \phi \right) i_\theta} \right] \nonumber\]

Also, in triangle OCO, \( OC = u_q = q + c \cot \phi \) and

\[\frac{R}{OC} = \frac{\sin \theta_q}{\sin (\theta_q + i_\theta)} = \sin \phi \nonumber\]

and in triangle OCP in Fig. 5 we have \( OC = u_q = p + c \cot \phi \).

\[\frac{R}{OC} = \frac{\sin \theta_q}{\sin (\theta_q + i_\theta)} = \sin \phi \nonumber\]

which results in

\[\delta = \sin^{-1} \left( \frac{\sin i_\theta}{\sin \phi} \right) - i_\theta \quad \beta = \sin^{-1} \left( \frac{\sin i_\theta}{\sin \phi} \right) + i_\theta \tag{23}\]
From equation 16 it is seen that

$$\frac{\partial \phi}{\partial \alpha} = e^{\tan \phi \cdot \alpha} = e^{-\tan \phi \cdot \frac{d\alpha}{\cos \phi \cdot \alpha}}$$

(21)

If we let $p = p + c \cot \phi$ and $q = q + c \cot \phi$

(22)

From triangles OCP and OCP (similar to Sekoloski (1963)) it is found that

$$q = \cos i_n \left( \cos i_n - \frac{\sin^2 \phi - \sin^2 i_n}{\sin^2 i_n} \right) u_n$$

(23)

$$p = \cos i_n \left( \cos i_n + \frac{\sin^2 \phi - \sin^2 i_n}{\sin^2 i_n} \right) u_n$$

If $N_q$ is defined as

$$N_q = \frac{p}{q} = \frac{p + c \cot \phi}{q + c \cot \phi}$$

(24)

then using eq. (23) and (21), $N_q$ is evaluated as

$$N_q = \frac{\cos i_n \left( \cos i_n + \frac{\sin^2 \phi - \sin^2 i_n}{\sin^2 i_n} \right) e^{\tan \phi \cdot \alpha}}{\cos i_n \left( \cos i_n - \frac{\sin^2 \phi - \sin^2 i_n}{\sin^2 i_n} \right) e^{-\tan \phi \cdot \frac{d\alpha}{\cos \phi \cdot \alpha}}}$$

(25)

If the usual form of $p = cN_q = qN_q$ is used, then it is clear from (24) that

$$p = N_q \left( q + c \cot \phi \cdot (N_q - 1) \right)$$

(26)

Thus

$$N_q = \frac{c \cot \phi}{c \cot \phi - 1}$$

(27)

Since $N_q$ is a function of $c/q$ it has to be evaluated by trial and error. Fig. 6 shows $N_q$ plotted as a function of $k/n$, the horizontal acceleration ratio, for $\phi$ from 1 to 45 degree for the case of $c/q = 0$ which corresponds to $c = 0.0$. The radial lines in Fig. 6 and 7 show the ratio of mobilisation of shear strength for surcharge region which varies between 0 to 1. (The ratio of mobilisation of shear strength is defined as $(q + \tan \phi)/(c + q + \tan \phi)$). Fig. 8 gives $N_q$ vs friction angle in degrees for various values of $k_0$ up to $0.5g$.

![Figure 6. Bearing capacity factor $N_q$ for $c = 0$.](image)

![Figure 7. Bearing capacity factor $N_q$ for $c = 0$, plotted vs $\phi$.](image)
Figure 8. $N_q$ for $c/q = 0, 0.1, 1, 10, 100$

Figure 9. Goursat zone

It is found that for $c = 0$, $N_q$ agrees with values obtained by Meyerhoff (1953), Sokolovski (1965) and Sansa and Iosifescu (1990). However for $c/q$ different from zero, different results are obtained. Fig. 8 presents $N_q$ in linear scale, for $c = 5, 10, 15, 20, 25$ degrees and for $c/q = 0, 0.1, 1, 10, 100$. It is clear that at larger $c$ values, an increase of $c/q$ (larger cohesion or smaller surcharge) results in increases in $N_q$ values. Curves for $N_q$ are not presented because equation (27) makes the evaluation easy.

**EVALUATION OF $N_q$**

For evaluation of $N_q$, integration of the equilibrium equation is carried out along ABCD. Since radial zero extension lines are straight, $\partial \phi / \partial x' = 0$ is equal to zero. If footing width is denoted by $b$, the geometric relations below follow (refer to Fig. 2):

$$OC = \frac{b \sin \alpha}{\cos \nu}$$

$$x = \frac{-b \sin \alpha}{\cos \nu} \sin (\alpha + \nu)$$

$$x = \frac{b \sin \alpha}{\cos \nu} \cos (\alpha + \nu)$$

$$OB = OC \ e^{\omega (\nu' - \nu)} = \frac{b \sin \alpha}{\cos \nu} \ e^{\omega \left( \frac{\nu' - \nu}{\nu} \right)}$$

$$x = \sin (\alpha + \nu) \frac{b \sin \alpha}{\cos \nu} \ e^{\omega \left( \frac{\nu' - \nu}{\nu} \right)}$$

$$x = \cos (\alpha + \nu) \frac{b \sin \alpha}{\cos \nu} \ e^{\omega \left( \frac{\nu' - \nu}{\nu} \right)}$$

$$x = OA = \frac{b \sin \alpha}{\sin \alpha} \ e^{\omega \left( \frac{\nu' - \nu}{\nu} \right)}$$

$$x = 0$$

Remembering that $Z = \gamma$ and $\lambda = \gamma \lambda$, then integration along AB, with $\theta = 0$, yields

$$u_{\theta} = u_{\alpha} + X \left[ \tan \phi (x - x_\theta) + 2 \lambda (x - x_\alpha) \right] - X Z \left[ \tan \phi (x - x_\alpha) + 2 \lambda (x - x_\alpha) \right]$$

where $u_{\alpha}$ and $u_{\theta}$ are reduced at A and B. Integration from B to C in the Goursat zone is carried out by using geometric relations from Fig. 9.

Let $\theta = 0 + \frac{\pi}{2} - \frac{\nu}{2}$

Then $\Omega = OB \ e^{\omega \left( \frac{\nu - 2\pi}{2} \right)} = OB \ e^{-\omega \left( \frac{\nu}{2} \right)}$

155
\[ x = \sqrt{R \cos(\theta)} - OB \ e^{-v(\theta - \phi)} \cos \theta \] (34)

\[ z = \sqrt{R \sin(\theta)} - OB \ e^{-v(\theta - \phi)} \sin \theta \] (35)

Furthermore,
\[ dx = OB \ e^{-v(\theta - \phi)} \left[ -\tan v \cos \theta - \sin \theta \right] \ d\theta \]

or
\[ dx = -OB \ e^{-v(\theta - \phi)} \sin (\theta + \mu) / \cos v \ d\theta \] (36)

Similarly,
\[ dz = OB \ e^{-v(\theta - \phi)} \cos (\theta + \mu) / \cos v \ d\theta \]

where \( \mu = \pi/4 + v/2 \).

By using equation (9) we get
\[ u_0 - 2 \tan \phi \eta = \frac{f}{\cos \eta} \left[ \frac{\left[ \tan \phi \cos (\theta + \mu) - \sin (\theta + \mu) \right]}{\cos \eta} \right] \ OB \ e^{-v(\theta - \phi)} \ d\theta \]

Noting that
\[ \int e^{\psi \cos (\alpha + \beta)} \ d\alpha = \cos \lambda \sin (\alpha + \beta - \lambda) \ e^{\psi \cos (\alpha + \beta)} \]

\[ \int e^{\psi \sin (\alpha + \beta)} \ d\alpha = -\cos \lambda \cos (\alpha + \beta - \lambda) \ e^{\psi \sin (\alpha + \beta)} \] (37)

where \( \tan \lambda = k \) after some manipulation and setting \( \lambda = 2 \tan \phi + \tan v \) we get
\[ u_0 = u_0 \ e^{-v(\theta - \phi)} + OB \ \frac{\cos \eta}{\cos \phi} \ e^{-v(\theta - \phi)} \]

\[ \left[ e^{+\psi} \left[ \tan \phi \sin \xi + \hat{\phi} \cos \xi \right] - Z \left( \tan \phi \cos \xi + \hat{\phi} \sin \xi \right) \right] \]

\[ -e^{-\psi} \left[ \tan \phi \sin \xi + \hat{\phi} \cos \xi - Z \left( \tan \phi \cos \xi + \hat{\phi} \sin \xi \right) \right] \] (38)

where \( \xi = \phi + \pi/4 + v/2 - \lambda \) and \( \xi = \phi + \pi/4 + v/2 - \lambda \).

Furthermore, integration along CD will result in
\[ u_{\infty} = u_{\infty} + X \phi \left[ \tan \phi (\eta_0 - \xi) + \hat{\phi} (\eta_0 - \xi) \right] \]

\[ -Z \hat{\phi} (\eta_0 - \xi) - \hat{\phi} (\eta_0 - \xi) \] (39)

To evaluate \( N_p \), we set \( u_0 = 0, \ c = \phi, \ X = k \ y \) and \( Z = v \). Because the pressure distribution under the footing is linear, we set \( p \) equal to the average pressure in the bearing capacity formula
\[ p = c N_p + q N_q + 0.5 yb N_q \] and use equation (23) to find \( N_p \). The result of computing \( N_p \), is shown in Fig. 30 for \( k = 45, 40, 35, 30, 27 \) degrees and for \( k \) ranging from zero to 0.5. The results are lower than those of Sarma and Kassif, although with the similar trend, and almost identical to Sokolowski's (1965) for \( k = c \).
Fig. 10 shows $N_s$ plotted as a function of $q$, logarithmic scale. It is found that the ratio of seismic bearing capacity factor $N_s$ to the static bearing capacity factor depends mainly on $k_s$ and does not depend strongly on friction angle $k$. This trend is shown in Fig. 12. This reduction curve can be used to calculate the seismic bearing capacity factor $N_s$ if static bearing capacity factor $N_s$ is evaluated by other methods. This trend compares favourably with the ratio evaluated by Rinaldi, Elms and Bushu (1999) in their Fig. 5(a).

Figure 10. Bearing capacity factor $N_s$ plotted against seismic coefficient $k_s$

Figure 11. Bearing capacity factor $N_s$

Figure 12. Seismic to static bearing capacity ratio

**CONCLUSIONS**

Based on the work presented in the paper the following conclusions can be made:

1. The method of zero extension line is capable of predicting seismic bearing capacity factors.

2. The cohesion, surcharge and gravity bearing capacity factors, $N_s$, $N_d$ and $N_c$, are reduced due to seismic acceleration, and inertial forces within the soil.

3. For $c = 0$, $N_s$ agrees with values obtained by Meyerhof (1955), Siscolovski (1965) and Sroka and Hascelik (1990). However, for $c 
eq 0$, different from zero, different values are obtained. In all cases $N_s = \cot \phi (N_s - 1)$.  

158
4. $N_v$ values are lower than Sarma and Jossifelis (1990) but have similar trends. $N_v$ values are identical to Sokolowski for $k_e = 0$.

5. The ratio of seismic bearing capacity factor $N_v$ over the static bearing capacity factor depends on $k_e$ and is not sensitive to value of $\phi$. This ratio by zero extension line theory compares favourably with the results of Richards, Elms and Budhu (1993).

ACKNOWLEDGEMENT

The authors acknowledge the support of the University of Canterbury and Shiraz University.

REFERENCES


10. Anvar, S A and Ghahramani, A 1995. Dynamic active earth pressure by zero extension line. *Proc. Third International Conf. on Recent Advances in Geotechnical Engineering and Soil Dynamics, St. Louis, Missouri, USA*.