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IMBEDDING METHOD FOR BEAM ON WINKLER FOUNDATION*

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Abstract - The problem of beam on Winkler foundation is usually solved by finite differences, the ACI recommended method for combined footings, and the Hetnyi solution. The proposed method uses an imbedding approach. The beam is divided into n nodes and the information on shear and moments at adjacent nodes are related through reactions and six imbedding coeffi - cients. Thus shear, moment and reaction at all nodes can be calculated by using a common desk calculator, sample examples are presented to illus - trate the method and comparison has been made with the finite difference, the ACI recommended method and the Hetnyi solution. A substantial num - ber of nodes can be handled easily with a desk calculator with negligible error of closure.

1. INTRODUCTION

The problem of beam on Winkler foundation is usually solved by several methods. The finite difference solution is one of the common methods for solving such problems [1,2]. In the following paper the imbedding method is applied to solve the problem of beam on Winkler foundation. The method makes possible the use of a desk calculator for solving the problem of the beam with several nodes. The method does not require the usual matrix inversion, which limits the capacity of common desk calculators. A sample problem with 28 nodes is solved, although many more nodes could be taken with the same desk calculator (usually 10 to 12 nodes are sufficient for most problems of this kind). Furthermore, for comparison, a problem is solved with finite difference (matrix inversion) and the imbedding method and very good results are obtained. A comparison is also made with the ACI suggested method and the Hetnyi method for combined footings.

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2. THEORY

Consider a beam on Winkler foundation as shown in Fig. 1. The beam is divided into (n − 1) equal sections with length ΔL and the loads are assumed to act at node 1 to n, designated by p_1, p_2, p_n. In the general case, the loads and moments can be considered to apply at any point and then be transferred to adjacent nodes, or the nodes can be selected at the applied load's location. For simplicity it is assumed that the width of the beam (B) and its moment

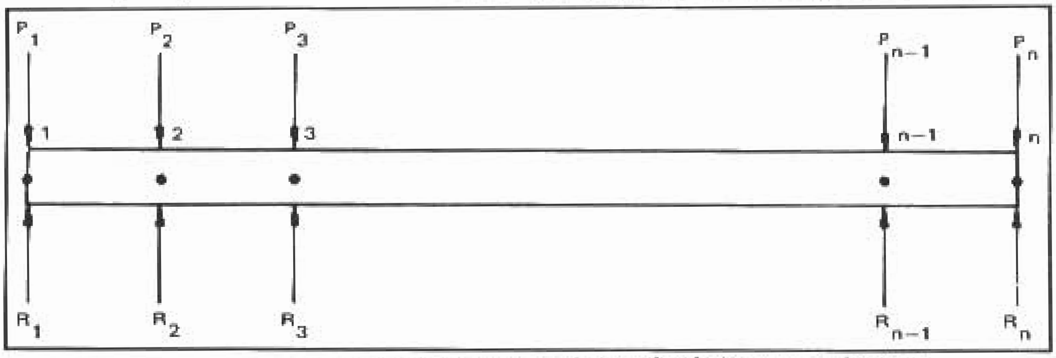

Fig. 1. A general beam on Winkler foundation divided into n-1 element.

of inertia I are constant. For Winkler foundation we know $p=KY$ where p is pressure transferred to soil foundation, Y is deflection and K is the modulus of subgrade reaction (3). If reactions are assumed to be $R_1, R_2, \ldots R_n$ acted at node 1, 2, ... and n, we will have $R_i=KY_i \Delta LB$ for any node and elements (Fig. 2), this relation is also true for the end nodes following recommendation by Bowels (1).

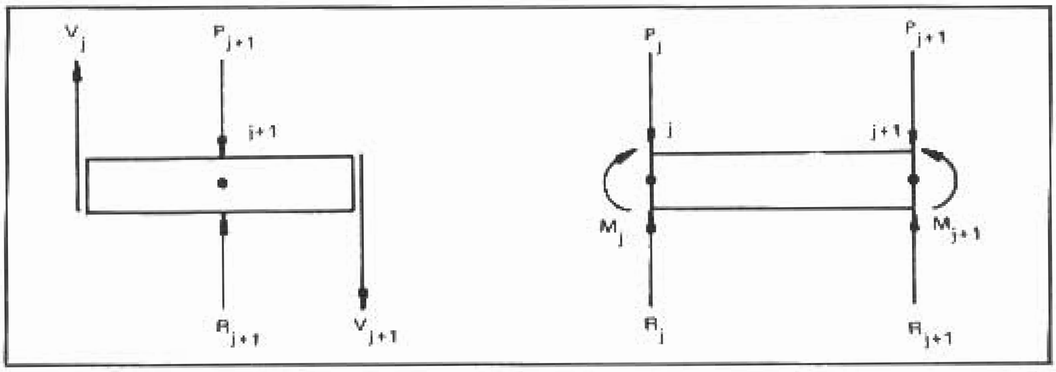

Fig. 2. Shear forces and moments on an element.

Applying the imbedding method we assume that

$$V_j = A_jR_j + B_jR_{j-1} + C_j$$

$$\frac{M_j}{\Delta L} = D_jR_j + E_jR_{j-1} + F_j$$

where V_j is the shear force at element j, M_j is moment at node j and A_j, B_j, C_j, D_j, E_j, F_j, are the coefficients which would be found later. The above equations show that the moment and the shear at each node (j) and element (j) are related to reactions at (j) and (j−1) th nodes by six coefficients A_j,... F_j. From Fig. (2) it is clear that

$$V_{j+1} = V_j + R_{j+1} - P_{j+1}$$

$$\frac{M_{j+1}}{\Delta L} = \frac{M_j}{\Delta L} + V_j$$

Using finite difference method in the equation of beam M=−EI Y'' we will have

$$M_j = -EI\left(\frac{Y_{j+1} + Y_{j-1} - 2Y_j}{\Delta L^2}\right)$$

but

$$Y_j = \frac{R_j}{K\Delta LB}$$

$$\therefore \frac{M_j}{\Delta L} = \frac{-EI}{(\Delta L)^4 KB}(R_{j-1} + R_{j+1} - 2R_j)$$

let

$$\alpha = \frac{\Delta L^4 KB}{EI}$$

then

$$\frac{M_j}{\Delta L} = -\frac{1}{\alpha}(R_{j-1} + R_{j+1} - 2R_j)$$

or

$$R_{j+1} = -\alpha\frac{M_j}{\Delta L} + 2R_j - R_{j-1}$$

therefore we will have the following equations

$$V_j = A_jR_j + B_jR_{j-1} + C_j \tag{1}$$

$$\frac{M_j}{\Delta L} = D_jR_j + E_jR_{j-1} + F_j \tag{2}$$

$$V_{j+1} = V_j + R_{j+1} - P_{j+1} \tag{3}$$

$$\frac{M_{j+1}}{\Delta L} = \frac{M_j}{\Delta L} + V_j \tag{4}$$

$$R_{j+1} = -\alpha \frac{M_j}{\Delta L} + 2R_j - R_{j-1} \tag{5}$$

if we substitute Eq. (3) in Eq. (1) we will get

$$A_{j+1}R_{j+1} + B_{j+1}R_j + C_{j+1} = A_jR_j + B_jR_{j-1} + C_j + R_{j+1} - P_{j+1}$$

or

$$(A_{j+1}-1)R_{j+1} + (B_{j+1}-A_j)R_j - B_jR_{j-1} + C_{j+1} - C_j + P_j+1 = 0$$

if we substitute R_{j+1} from Eq. (5) in the above relation we will have

$$(A_{j+1} - 1)(-\alpha \frac{M_j}{\Delta L} + 2R_j - R_{j-1}) + (B_{j+1} - A_j)R_j - B_jR_{j-1} + C_{j+1} - C_j + P_{j+1} = 0$$

substitute the value of $\frac{M_j}{\Delta L}$ from Eq. (2) we get

$$(A_{j+1} - 1)(-\alpha D_jR_j - \alpha E_jR_{j-1} - \alpha F_j + 2R_j - R_{j-1}) + (R_{j+1} - A_j)R_j - B_jR_{j-1} + C_{j+1} - C_j + P_{j+1} = 0$$

or

$$R_j[2(A_{j+1} - 1) - \alpha D_j(A_{j+1} - 1) + (B_{j+1} - A_j)] - R_{j-1}[(A_{j+1} - 1) + \alpha E_j(A_{j+1}) + B_j] - (A_{j+1} - 1)\alpha F_j + C_{j+1} - C_j + P_{j+1} = 0$$

The above relation is true for any value of R_j and R_{j-1}. Therefore the coefficients of R and the constant value have to be zero; it means

$$2(A_{j+1} - 1) - \alpha D_j(A_{j+1} - 1) + B_{j+1} - A_j = 0 \tag{6}$$

$$(A_{j+1} - 1) + \alpha E_j(A_{j+1} - 1) + B_j = 0 \tag{7}$$

$$-(A_{j+1} - 1)\alpha F_j + C_{j+1} - C_j + P_{j+1} = 0 \tag{8}$$

In a similar procedure by substituting Eq. (2) in (4) and using Eq. (5) we will have

$$2D_{j+1} - \alpha D_{j+1} D_j + E_{j+1} - D_j - A_j = 0 \tag{9}$$

$$D_{j+1} + \alpha E_j \; D_{j+1} + E_j + B_j = 0 \tag{10}$$

$$-D_{j+1} \alpha F_j + F_{j+1} - F_j - C_j = 0 \tag{11}$$

Finally from the above equation (Eq. 6 through 11) we get

$$A_{j+1} = 1 - \frac{B_j}{1 + \alpha E_j} \tag{12}$$

$$B_{j+1} = A_j + (2 - \alpha D_j) \frac{B_j}{1 + \alpha E_j} \tag{13}$$

$$C_{j+1} = C_j - P_{j+1} - \alpha F_j \frac{B_j}{1 + \alpha E_j} \tag{14}$$

$$D_{j+1} = - \frac{E_j + B_j}{1 + \alpha E_j} \tag{15}$$

$$E_{j+1} = (2 - \alpha D_j) \frac{E_j + B_j}{1 + \alpha E_j} + D_j + A_j \tag{16}$$

$$F_{j+1} = -\alpha F_j \frac{E_j + B_j}{1 + \alpha E_j} + F_j + C_j \tag{17}$$

Thus the coefficients at node j+1 are evaluated from node j by using the above relations. At node one we know that

$$M_1 = 0$$

and

$$V_1 = R_1 - P_1$$

therefore from Eq. (1) and (2) we get

$$A_1 = 1 \quad B_1 = 0 \quad C_1 = -P_1 \quad D_1 = 0 \quad E_1 = 0 \quad F_1 = 0$$

Then the coefficient at node (2) can be evaluated trom the coefficient at node (1) and successively these coefficients would be found at each node. As it can be seen, the above coefficients are of order ΔL^4 (α is of order ΔL^4), which means that the error would be of order ΔL^4.

Evaluation of the above coefficient can be done by an ordinary desk calculator. In other words, instead of inversion, an n x n matrix, the above coefficients are found sequentially n times from known values of $A_1, B_1, ..., F_1$.

At the last node again we know that $M_n = 0$ and from the definition of V_n, which is the shear at nth element, knowing that for n nodes we have n−1 element we can write

$$V_n = 0$$

$$\therefore A_n R_n + B_n R_{n-1} + C_n = 0$$

$$D_n R_n + E_n R_{n-1} + F_n = 0$$

where A_n, B_n, C_n, D_n, E_n and F_n are known values. Therefore R_{n-1} and R_n can be calculated as below

$$R_n = \frac{\begin{vmatrix} -C_n & B_n \\ -F_n & E_n \end{vmatrix}}{\begin{vmatrix} A_n & B_n \\ D_n & E_n \end{vmatrix}} \quad (18.1) \qquad R_{n-1} = \frac{\begin{vmatrix} A_n & -C_n \\ D_n & -F_n \end{vmatrix}}{\begin{vmatrix} A_n & B_n \\ D_n & E_n \end{vmatrix}} \quad (18.2)$$

knowing R_n and R_{n-1}, two different procedures can be used to calculate V, M, and R at each node.

Procedure (1).

Using Eqs. (3, 4) at (5) we can write

$$V_j = V_{j+1} - R_{j+1} + P_{j+1}$$

$$\frac{M_j}{\Delta L} = \frac{M_{j+1}}{\Delta L} - V_j$$

$$R_{j-1} = 2R_j - R_{j+1} - \alpha \frac{M_j}{\Delta L}$$

Because $V_n = 0$, $M_n = 0$, R_n and R_{n-1} are known from Eq. (18.1) and (18.2) then V_{n-1}, M_{n-1} and R_{n-2} can be calculated from Eqs. (3 - 5).

However, it is preferable to once again use the imbedding approach. For simplicity the nodes are renumbered from left to right, starting from (1) (Fig.

3). Again $A_1 = 1$, $B_1 = 0$, $C_1 = -P_1$, $D_1 = 0$, $E_1 = 0$, $F_1 = 0$, and R_1 and R_2 are known, then A_2, B_2, C_2, D_2, E_2, F_2 can be found from Eqs. (12) through (17). Then V_2, M_2 and R_3 can be calculated using Eqs. (1), (2) and (5). This procedure is extended to Vn and M_n which gives the errors of closure.

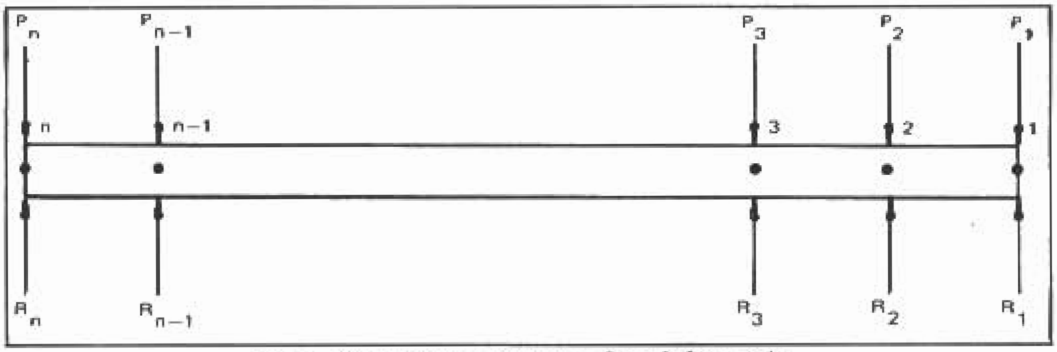

Fig. 3. Renumbering the beam from left to right.

Therefore, by this method, R_j and M_j can be calculated easily without any matrix inversion with a simple calculator.

Sample example 1.

A beam of length L = 88.58 ft(27 m), thickness D = 3.44 ft(1.05 m) and width B = 9.84 ft(3 m) is subjected to loads, as shown in Fig. (4). K = 54.19 (1b/cf) (1.5 Kg/cm^3) and E = 29.87 x 10^5 (1b/in^2) (210000 Kg/cm^2). The reactions, shear forces and moments are found by the imbedding method.

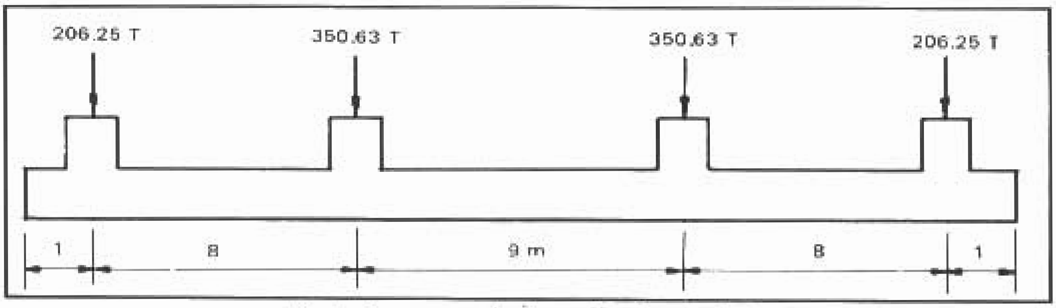

Fig. 4. Forces on the beam for Example 1.

Solution.

The beam is divided into 27 elements (28 nodes), each one a meter apart (Fig. 5).

$$P_2 = 206.25T,\ P_{10} = 350.63T,\ P_{19} = 350.63T,\ P_{27} = 206.25T$$

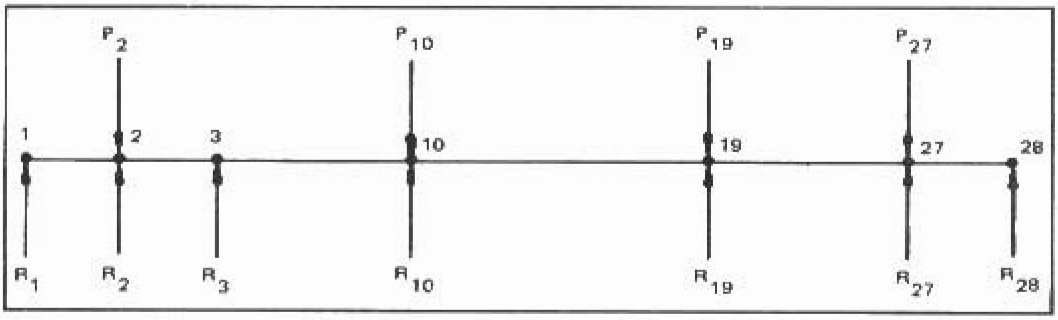

Fig. 5. Division of the beam in the 27 elements.

$$I = \frac{1}{12} BD^3 = \frac{1}{12} (300)\ (105)^3 = 28940625 \text{ cm}^4$$

Knowing

$$A_1 = 1,\ B_1 = 0,\ C_1 = 0,\ D_1 = 0,\ E_1 = 0,\ F_1 = 0$$

then

$$A_2 = 1 - \frac{B_1}{1 + \alpha E_1} = 1$$

$$B_2 = A_1 + (2 - \alpha D_1) \frac{B_1}{1 + \alpha E_1} = 1$$

$$C_2 = C_1 - P_2 - \alpha F_1 \frac{B_1}{1 + \alpha E_1} = -206250 \quad \text{kg}$$

$$D_2 = - \frac{E_1 + B_1}{1 + \alpha E_1} = 0$$

$$E_2 = (2 - \alpha D_1) \frac{E_1 + B_1}{1 + \alpha E_1} + D_1 + A_1 = 1$$

$$F_2 = -\alpha F_1 \frac{E_1 + B_1}{1 + \alpha E_1} + F_1 + C_1 = 0$$

Knowing the six coefficients $A_2, \ldots, F_2$ then they can be calculated at the next step and so on and finally at node 28 $A = -8.4371$, $B = 14.31$, $C = -242464.86$, $D = -55.4268$, $E = 69.7368$ and $F = -485645.191$. It is clear that, with a programable calculator, the above calculation can be easily done. Knowing that shear and moment at node 28 is zero then R at node 28 and 27 can be found from Equation (18) and (2)

$$R_{27} = 45618.45 \text{ kg}, \qquad R_{28} = 48633.97 \text{ kg}$$

Renumbering nodes from left to right (Fig. 6) and knowing

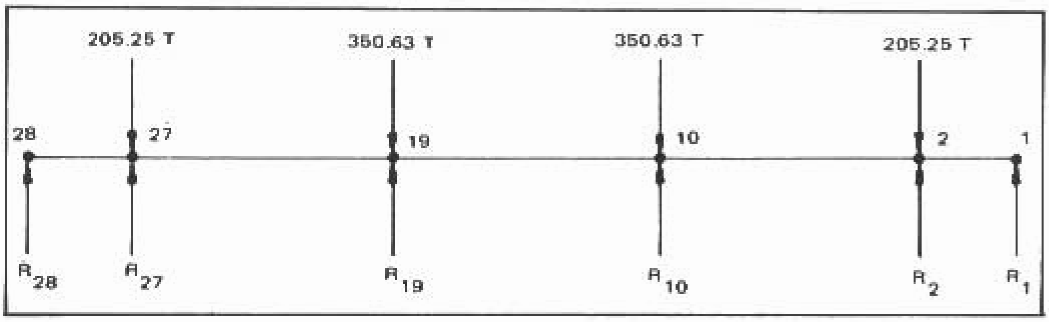

Fig. 6. Renumbering the beam from right to left.

$$A_1 = 1,\ B_1 = 0,\ C_1 = 0,\ D_1 = 0,\ E_1 = 0,\ F_1 = 0$$

$$R_1 = 48633.97 \text{ kg} \qquad R_2 = 45618.45 \text{ kg}$$

we would have

$$A_2 = 1,\ B_2 = 1,\ C_2 = -206250,\ D_2 = 0,\ E_2 = 1,\ F_2 = 0$$

and

$$V_2 = A_2R_2 + B_2R_1 + C_2 \qquad V_2 = -111997.58 \text{ kg}$$

$$\frac{M_2}{\Delta L}\ D_2R_2 + E_2R_1 + F_2 \qquad \frac{M_2}{\Delta L} = 48633.97 \text{ kg}$$

$$R_3 = 2R_2 - R_1 - \alpha \frac{M_2}{\Delta L} \qquad R_3 = 42242.82 \text{ kg}$$

The above procedure is continued untill V_{28} and M_{28} are calculated, which are the errors of closure. Table 1 gives the complete result for the sample problem and it is clear that the error of closure is negligible in comparison with the magnitude of moments and shears.

Table 1. Shear and mount at nodes by the imbedding method.

Node #	V Kg	M Kgm	R Kg
1	48633.97	0	48633.98
2	−111997.58	48633.97	48633.97
3	−69754.75	−63363.60	45618.45
4	−30418.39	−133118.36	42242.82
5	6997.17	−163536.74	39336.36
6	43702.79	−156539.58	37415.55
7	80857.55	−112836.79	36705.62
8	119296.93	−31979.24	37154.76
9	159257.72	87317.69	38439.38
10	−150536.62	246575.41	39960.78
11	−110651.81	96038.78	40835.66
12	−72428.95	−14613.03	39884.81
13	−35759.85	−87041.99	38222.86
14	-1.8×10^{-3}	−122801.83	36669.11
15	35759.84	−122801.83	35759.84
16	72428.95	−87041.99	35759.84
17	110651.81	−14613.04	36669.11
18	150536.62	96038.77	38222.86
19	−159257.72	246575.39	39884.81
20	−119296.94	87317.67	40835.66
21	−80857.55	−31979.26	39960.78
22	−43702.79	−112836.82	38439.38
23	−6997.16	156539.61	37154.76
24	30418.39	−163536.77	36705.63
25	69754.76	−133118.38	37415.56
26	111997.59	−63363.61	39336.37
27	−48663.95	48633.98	42242.83
28	3.1600×10^{-2}*	3.12×10^{-2}*	45618.48

* Closure Error
1 Kg = 2.2 lb
1 Kgm = 7.217 ft. lb

Sample example 2.

In the following beam Fig. 7 E = 3250 ksi (2.285 x 10^5 Kg/cm^2), K = 48 K/cf (.77 Kg/cm^3) and width B = 10 ft (3.048 m); for comparison this problem has been solved by finite difference [1] and by the imbedding method. In both methods the beam is divided into ten equal sections (Fig. 7). Table 2 shows the shear and bending moment at the nodes. As it can be seen, the moment should be symmetric at the nodes. The symmetry is held in the im - bedding method, while in the finite difference method, the symmetry is not held for the moment at symmetric nodes.

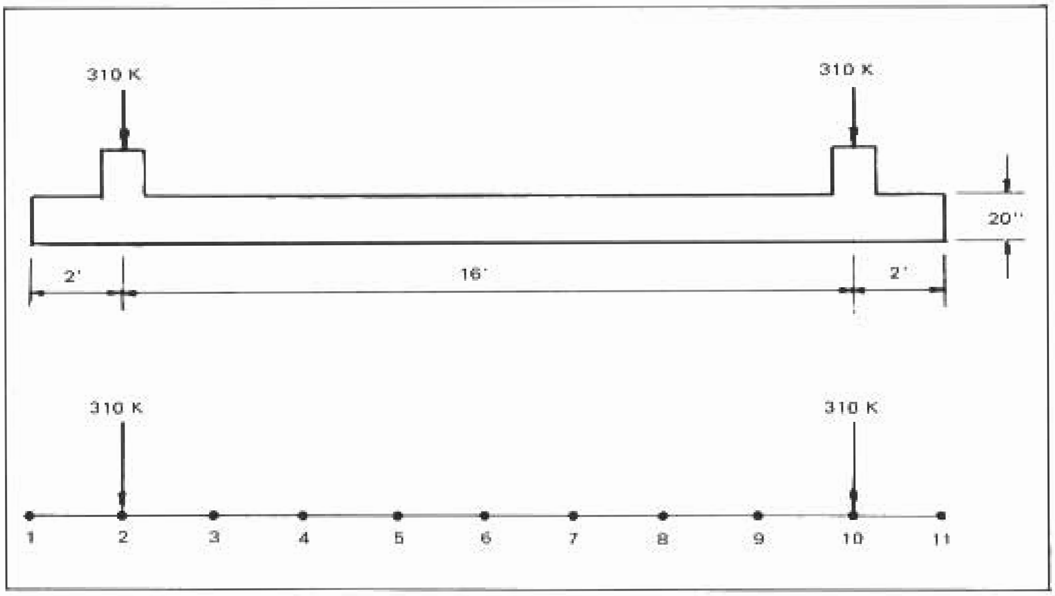

Fig. 7. Forces on the beam for Example 2.

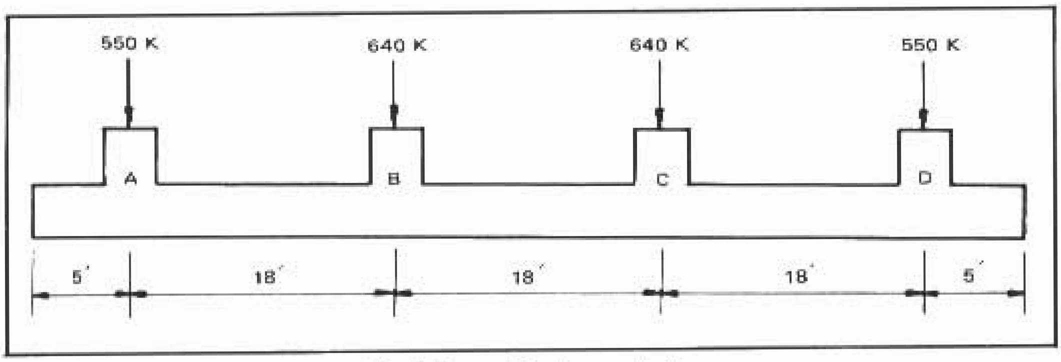

Fig. 8. Beam for Example 3.

Sample example 3.

The beam of Fig. (8) with E = 432 x 10^3 Kips/ft^2 (2.11x10^5 Kg/cm^2), I = 5.0625 ft^4 (4.37x10^6 cm^4), K = 200 kip/ft^3 (3.2 Kg/cm^3) and width B = 18 ft (5.49 m) is considered. The moments and shears are calculated at columns by

(a) Imbedding method with 64 elements, each 1 foot in length.

(b) Method recommended for the solution of combined footings by ACI [3] which is based on the work of Kramerisch and Rogers [4].

Table 2. Comparison of the imbedding and finite difference methods.

Node #	Imbedding Method			Finite Difference		
	V K_{ip}	M ft $-K_{ip}$	R K_{ip}	V K_{ip}	M ft $-K_{ip}$	R K_{ip}
1		0	63.6997		0	63.7
	63.6997			63.70		
2		127.3994	60.2306		127.4	60.2
	186.0697			186.07		
3		244.7398	56.4904		244.76	56.5
	129.5793			129.58		
4		503.8984	53.2708		503.91	53.3
	76.3084			76.31		
5		656.5154	51.1229		656.52	51.1
	25.1856			25.19		
6		706.8864	50.3712		706.90	50.4
	25.1856			25.19		
7		656.5154	51.1229		656.56	51.1
	76.3084			76.31		
8		503.8984	53.2708		503.90	53.3
	129.5795			129.58		
9		244.7398	56.4904		244.75	56.5
	186.0697			186.07		
10		127.3994	60.2306		127.43	60.2
	63.6997			63.7		
11		8×10^{-7}*	63.6997		0	63.7
	$-4.6^{*}\times10^{-6}$					

* Closure Error

1 Kip = 453.59 Kg

1 ft - Kip = 138.25 Kgm

(c) For comparison, the problem is solved by the Hetnyi procedure of beam on elastic foundation. The results are shown in Table 3. The results show that the imbedding method is not only easy to use but is also sufficiently accurate.

Table 3. Comparison of the imbedding ACI and Hetnyi methods.

Column	Imbedding meth.	ACI	Hetnyi
Point	M $K_{ip.ft}$	M	M
A	687.00	560.	637.74
B	906.960	870.	913.00
C	906.960	870.	913.00
D	687.00	560.	637.74

1 Kg = 2.2 lb
1 Kg = 7.217 ft.lb

3. DISCUSSION

Considering the theoretical development and sample problems which are solved, it is seen that the imbedding method for solving the problems of beam on Winkler foundation is capable of being solved with a substantial number of nodes and negligible closure errors. In particular, it is seen that the symmetry in the second example is preserved especially in moments by the imbedding method, while the finite difference method has a slight nonsymmetry. It is clear that the imbedding method can be used with common desk calculators while the matrix inversion needs a computer. It should be mentioned that the imbedding method can also be applied to the solution of lateral loaded piles and sheet piles. Furthermore, variation of moment of inertia and existence of external moment can easily be incorporated in the calculation procedure by changing and modifying Eq. (4).

4. CONCLUSION

It can be concluded that the imbedding method is capable of solving the problem of beam on Winkler foundation by hand or common desk calcu -

lators and with only small errors of closure. It is further demonstrated that the imbedding method, in comparison with the methods of finite differences, that method as recommended by ACI and also by the Hetnyi solution, has shown very accurate and promising results.

NOMENCLATURE

$A_j, B_j, C_j, D_j, E_j, F_j$	Imbedding coefficients
B	Width of beam
D	Depth of beam
E	Elastic modulus of concrete
I	Moment of inertia of beam
K	Modulus of subgrade reaction
L	Length of beam
M_j, M	Moment
n	Number of nodes
P_i, P	Load
p	Pressure
R_j, R	Reaction
V_j, V	Shear force
Y_j, Y	Deflection
Y''	Second derivative of deflection
α	Parameter in moment equation
ΔL	Length of each element

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