IMBEDDING METHOD FOR BEAM ON WINKLER FOUNDATION

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Abstract. The problem of beams on Winkler foundation is usually solved by
finite differences, the ACI recommended method for combined footings,
and the Hetenyi solution. The proposed method uses an imbedding approach.
The beam is divided into a nodes and the information on shear and moments
at adjacent nodes are related through tractions and six imbedding coeffi-
cients. Thus shear, moment and reaction at all nodes can be calculated by
using a common desk calculator, sample examples are presented to illus-
trate the method and comparison has been made with the finite difference,
the ACI recommended method and the Hetenyi solution. A substantial num-
ber of nodes can be handled easily with a desk calculator with negligible
error of closure.

1. INTRODUCTION

The problem of beam on Winkler foundation is usually solved by several
methods; The finite difference solution is one of the common methods for
solving such problems [1,2]. In the following paper the imbedding method is
applied to solve the problem of beam on Winkler foundation. The method
makes possible the use of a desk calculator for solving the problem of the beam
with several nodes. The method does not require the usual matrix inversion,
which limits the capacity of common desk calculators. A sample problem with
28 nodes is solved, although many more nodes could be taken with the same
desk calculator (usually 10 to 12 nodes are sufficient for most problems of this
kind). Furthermore, for comparison, a problem is solved with finite difference
(matrix inversion) and the imbedding method and very good results are obtained.
A comparison is also made with the ACI suggested method and the Hetenyi
method for combined footings.

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2. THEORY

Consider a beam on Winkler foundation as shown in Fig. 1. The beam is divided into \((n-1)\) equal sections with length \(\Delta L\) and the loads are assumed to act at node 1 to \(n\) designated by \(P_1, P_2, P_n\). In the general case, the loads and moments can be considered to apply at any point and then be transferred to adjacent nodes, or the nodes can be selected at the applied load's location.

For simplicity it is assumed that the width of the beam \((b)\) and its moment of inertia \(I\) are constant. For Winkler foundation we know \(p = kY\) where \(p\) is pressure transferred to soil foundation, \(Y\) is deflection and \(k\) is the modulus of subgrade reaction \((3)\). If reactions are assumed to be \(R_1, R_2, \ldots, R_n\) acted at node 1, 2, \ldots and \(n\), we will have \(R_j = kY_j\) \(\Delta L\) for any node and element (Fig. 2), this relation is also true for the end nodes following recommendation by Bowes (1).

![Fig. 1. A general beam on Winkler foundation divided into \(n\) elements.](image)

![Fig. 2. Shear forces and moments on an element.](image)

Applying the imbedding method we assume that
\[ V_j = A_j R_j + B_j R_{j-1} + C_j \]
\[ M_j = \frac{D_j R_j + E_j R_{j-1} + F_j}{\Delta L} \]

where \( V_j \) is the shear force at element \( j \), \( M_j \) is the moment at node \( j \) and \( A_j, B_j, C_j, D_j, E_j, F_j \) are the coefficients which would be found later. The above equations show that the moment and the shear at each node \( j \) and element \( j \) are related to reactions at \( j \) and \( (j-1) \) th nodes by six coefficients \( A_{j-1} \), \( B_{j-1} \), \( C_j \), \( D_{j-1} \), \( E_j \), \( F_j \). From Fig. (3) it is clear that

\[ V_{j+1} = V_j + R_{j+1} - P_{j+1} \]
\[ M_{j+1} = \frac{M_j}{\Delta L} + V_j \]

Using finite difference method in the equation of beam \( M = -EI V'' \) we will have

\[ M_j = -EI \left( \frac{Y_{j+1} + Y_{j-1} - 2Y_j}{\Delta L^2} \right) \]

but

\[ Y_j = \frac{R_j}{K_\Delta L} \]

\[ \frac{M_j}{\Delta L} = -\frac{EI}{(\Delta L)^2} \left( R_{j-1} + R_{j+1} - 2R_j \right) \]

let

\[ \alpha = \frac{\Delta L^4 K_B}{EI} \]

then

\[ \frac{M_j}{\Delta L} = -\frac{1}{\alpha} \left( R_{j-1} + R_{j+1} - 2R_j \right) \]

or

\[ R_{j+1} = -\alpha \frac{M_j}{\Delta L} + 2R_j - R_{j-1} \]

therefore we will have the following equations

\[ V_j = A_j R_j + B_j R_{j-1} + C_j \]  \hspace{1cm} (1)
\[ \frac{M_j}{\Delta L} = D_j R_j + E_j R_{j-1} + F_j \]  \hspace{1cm} (2)
\[ V_{j+1} = V_j + R_{j+1} - P_{j+1} \quad (3) \]

\[ \frac{M_j}{\Delta L} = \frac{M_j}{\Delta L} + V_j \quad (4) \]

\[ R_{j+1} = \frac{\Delta L}{M_j} + 2R_j - R_{j-1} \quad (5) \]

If we substitute Eq. (3) in Eq. (1), we will get

\[ A_{j+1} + B_{j+1} R_j + C_{j+1} = A_j R_j + B_j R_{j-1} + C_j + R_{j+1} - P_{j+1} \]

or

\[ (A_{j+1} - 1) R_{j+1} + (B_{j+1} - A_j) R_j - B_j R_{j-1} + C_{j+1} - C_j + P_{j+1} = 0 \]

If we substitute \( R_{j+1} \) from Eq. (5) in the above relation we will have

\[ (A_{j+1} - 1)(-\frac{M_j}{\Delta L} + 2R_j - R_{j-1}) + (B_{j+1} - A_j) R_j - B_j R_{j-1} + C_{j+1} - C_j + P_{j+1} = 0 \]

Substitute the value of \( \frac{M_j}{\Delta L} \) from Eq. (2) we get

\[ \langle A_j + 1 \rangle + \langle A_{j+1} - 1 \rangle - \alpha D_j R_j - \alpha E_j R_{j-1} - \alpha F_j + 2R_j - R_{j-1} \rangle + \]

\[ \langle R_{j+1} - A_j R_j - B_j R_{j-1} + C_{j+1} - C_j + P_{j+1} \rangle = 0 \]

or

\[ R_j [2(A_{j+1} - 1) - \alpha D_j (A_{j+1} - 1) + (B_{j+1} - A_j)] - R_{j-1} (A_{j+1} - 1) \]

\[ + \alpha E_j (A_{j+1} - 1) + B_j - (A_j + 1 - 1) - \alpha F_j + C_{j+1} - C_j + P_{j+1} = 0 \]

The above relation is true for any value of \( R_j \) and \( R_{j-1} \). Therefore the coefficients of \( R \) and the constant value have to be zero; it means

\[ 2(A_{j+1} - 1) - \alpha D_j (A_{j+1} - 1) + B_{j+1} - A_j = 0 \quad (6) \]

\[ (A_{j+1} - 1) + \alpha E_j (A_{j+1} - 1) + B_j = 0 \quad (7) \]

\[ -(A_j + 1 - 1) - \alpha F_j + C_{j+1} - C_j + P_{j+1} = 0 \quad (8) \]

In a similar procedure by substituting Eq. (2) in (4) and using Eq. (5) we will have

\[ 2D_j + 1 - \alpha D_j + 1 + E_j + 1 - D_j - A_j = 0 \quad (9) \]
\[ D_{j+1} + \alpha E_j L_{j+1} E_j + B_j = 0 \]  
(19)

\[-D_{j+1} + \alpha F_{j+1} F_{j+1} - F_j - C_j = 0 \]  
(11)

Finally from the above equation (Eq. 6 through 11) we get

\[ A_{j+1} = 1 - \frac{B_j}{1 + \alpha E_j} \]  
(12)

\[ B_{j+1} = A_j + (2 - \alpha D_j) \frac{B_j}{1 + \alpha E_j} \]  
(13)

\[ C_{j+1} = C_j - F_{j+1} - \alpha F_j \frac{B_j}{1 + \alpha E_j} \]  
(14)

\[ D_{j+1} = \frac{E_j + B_j}{1 + \alpha E_j} \]  
(15)

\[ E_{j+1} = (2 - \alpha D_j) \frac{E_j + B_j}{1 + \alpha E_j} + D_j + A_j \]  
(16)

\[ F_{j+1} = -\alpha F_{j+1} \frac{E_j + B_j}{1 + \alpha E_j} + F_j + C_j \]  
(17)

Thus the coefficients at node \( j+1 \) are evaluated from node \( j \) by using the above relations. At node one we know that

\[ M_1 = 0 \]

and

\[ V_1 = -R_1 + F_1 \]

therefore from Eq. (1) and (2) we get

\[ A_1 = 1 \quad B_1 = 0 \quad C_1 = -P_1 \quad D_1 = 0 \quad E_1 = 0 \quad F_1 = 0 \]

Then the coefficient at node (2) can be evaluated from the coefficient at node (1) and successively these coefficients would be found at each node. As it can be seen, the above coefficients are of order \( \Delta L^4 \) (as is of order \( \Delta L^4 \)), which means that the error would be of order \( \Delta L^4 \).

Evaluation of the above coefficients can be done by an ordinary desk cal.
culator. In other words, instead of inversion, an \( n \times n \) matrix, the above coefficients are found sequentially \( n \) times from known values of \( A_1, B_1, \ldots, F_1 \).

At the last node again we know that \( M_{n} = 0 \) and from the definition of \( V_n \), which is the shear at \( n \)th element, knowing that for \( n \) nodes we have \( n - 1 \) element we can write

\[
V_n = 0
\]

\[
A_n R_n + B_n R_{n-1} + C_n = 0
\]

\[
D_n R_n + E_n R_{n-1} + F_n = 0
\]

where \( A_n, B_n, C_n, D_n, E_n \) and \( F_n \) are known values. Therefore \( R_{n-1} \) and \( R_n \) can be calculated as below

\[
R_{n} = \begin{bmatrix}
-A_n & B_n \\
-D_n & E_n
\end{bmatrix} \begin{bmatrix}
-R_{n-1} \\
-E_{n}
\end{bmatrix}
\]

\[
R_{n-1} = \begin{bmatrix}
-A_n & -C_n \\
-D_n & -E_n
\end{bmatrix} \begin{bmatrix}
R_n \\
F_n
\end{bmatrix}
\]

knowing \( R_n \) and \( R_{n-1} \), two different procedures can be used to calculate \( V, M \) and \( R \) at each node.

Procedure (1).

Using Eqs. (3, 4) at (5) we can write

\[
V_j = V_{j+1} - R_j + \frac{P_{j+1}}{\Delta L}
\]

\[
M_j = M_{j+1} - \frac{R_j}{\Delta L}
\]

\[
R_{j-1} = 2R_j - R_{j+1} - \frac{M_j}{\Delta L}
\]

Because \( V_n = 0, M_n = 0, R_n \) and \( R_{n-1} \) are known from Eq. (18.1) and (18.2) then \( V_{n-1}, M_{n-1} \) and \( R_{n-2} \) can be calculated from Eqs. (3-5).

However, it is preferable to once again use the embedding approach. For simplicity the nodes are renumbered from left to right, starting from (1) (Fig.
3). Again $A_1 = 1, B_1 = 0, C_1 = -P_1, D_1 = 0, E_1 = 0, F_1 = 0$, and $R_1$ and $R_2$ are known, then $A_2, B_2, C_2, D_2, E_2, F_2$ can be found from Eqs. (12) through (17). Then $V_2, M_2$ and $R_3$ can be calculated using Eqs. (4), (2) and (5). This procedure is extended to $V_n$ and $M_n$ which gives the errors of closure.

![Fig. 3. Modeling the beam from left to right.](image)

Therefore, by this method, $R_j$ and $M_j$ can be calculated easily without any matrix inversion with a simple calculator.

**Sample example 1.**

A beam of length $L = 88.58$ ft (27 m), thickness $D = 3.44$ ft (1.05 m) and width $B = 9.64$ ft (3 m) is subjected to loads, as shown in Fig. (4), $K = 54.19$ (lb/ft) (1.5 Kg/cm\(^2\)) and $E = 29.83 \times 10^6$ (lb/in\(^2\)) (210,900 Kg/cm\(^2\)). The reactions, shear forces and moments are found by the imbedding method.

![Fig. 4. Forces on the beam for Example 1.](image)

**Solution.**

The beam is divided into 27 elements (28 nodes), each one a meter apart (Fig. 5).

$$P_2 = 206.25\, \text{ST}, P_{16} = 350.63\, \text{ST}, P_{19} = 350.63\, \text{ST}, P_{27} = 206.25\, \text{ST}$$
Fig. 5. Division of the beam in the 27 elements.

\[ I = \frac{1}{12} BD^3 = \frac{1}{12} (300) (105)^3 = 28940625 \text{ cm}^4 \]

Knowing

\[ A_1 = 1, \; B_1 = 0, \; C_1 = 0, \; D_1 = 0, \; E_1 = 0, \; F_1 = 0 \]

then

\[ A_2 = 1 - \frac{B_2}{1 + aE_1} = 1 \]

\[ B_2 = A_2 + (2 - aD_1) \frac{B_1}{1 + aE_1} = 1 \]

\[ C_2 = C_1 - F_2 - aE_1 - \frac{B_1}{1 + aE_1} = -206250 \text{ kg} \]

\[ D_2 = - \frac{E_1 + B_1}{1 + aE_1} = 0 \]

\[ E_2 = (2 - aD_1) \frac{E_1 + B_1}{1 + aE_1} + D_1 + A_1 = 1 \]

\[ F_2 = -aE_1 - F_1 + B_1 + E_1 + C_2 = 0 \]

Knowing the six coefficients \( A_2, ..., F_2 \) then they can be calculated at the next step and so on and finally at node 28 \( A = -8.4371, B = 14.31, C = -242464.86 \), \( D = -15.4265, E = 69.7366 \) and \( F = -485645.191 \). It is clear that, with a
progranunable calculator, the above calculation can be easily done. Knowing
that shear and moment at node 28 is zero then R at node 28 and 27 can be
found from Equation (18) and (2)

\[ R_{27} = 45618.45 \text{ kg} \quad R_{28} = 48633.97 \text{ kg} \]

Renumbering nodes from left to right (Fig. 6) and knowing

\[ A_1 = 1, \quad B_1 = 0, \quad C_1 = 0, \quad D_1 = 0, \quad E_1 = 0, \quad F_1 = 0 \]

\[ R_1 = 48633.97 \text{ kg} \quad R_2 = 45618.45 \text{ kg} \]

we would have

\[ A_2 = 1, \quad B_2 = 1, \quad C_2 = -206250, \quad D_2 = 0, \quad E_2 = 1, \quad F_2 = 0 \]

and

\[ V_2 = -\frac{M_2}{\Delta L} + B_2 R_1 C_2 \quad V_2 = -111997.58 \text{ kg} \]

\[ R_3 = \frac{2M_2}{\Delta L} - R_1 - \frac{M_2}{\Delta L} \quad R_3 = 42242.82 \text{ kg} \]

The above procedure is continued until \( V_{28} \) and \( M_{28} \) are calculated, which
are the errors of closure. Table 1 gives the complete result for the sample
problem and it is clear that the error of closure is negligible in comparison with
the magnitude of moments and shears.

Table 1. Shear and moment at nodes by the imbedding method.

<table>
<thead>
<tr>
<th>Node #</th>
<th>V Kg</th>
<th>M Kgm</th>
<th>R Kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>48633.97</td>
<td>0</td>
<td>48633.97</td>
</tr>
<tr>
<td>2</td>
<td>-111997.58</td>
<td>48633.97</td>
<td>48633.97</td>
</tr>
<tr>
<td>3</td>
<td>-69754.75</td>
<td>-5336.69</td>
<td>45618.45</td>
</tr>
<tr>
<td>4</td>
<td>-50418.39</td>
<td>-133118.36</td>
<td>42242.02</td>
</tr>
<tr>
<td>5</td>
<td>65997.17</td>
<td>-163536.74</td>
<td>39336.36</td>
</tr>
<tr>
<td>6</td>
<td>43702.79</td>
<td>-136535.58</td>
<td>37415.55</td>
</tr>
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<td>80852.55</td>
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<td>36705.62</td>
</tr>
<tr>
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<td>119296.93</td>
<td>-31979.24</td>
<td>37154.76</td>
</tr>
<tr>
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<td>159257.72</td>
<td>87317.69</td>
<td>38439.38</td>
</tr>
<tr>
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<td>-159536.62</td>
<td>246575.41</td>
<td>39960.78</td>
</tr>
<tr>
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<td>-110651.81</td>
<td>96038.78</td>
<td>40835.66</td>
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<td>-357041.99</td>
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<td>35759.64</td>
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<td>-146130.53</td>
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<td>159257.72</td>
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<td>-636536.61</td>
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<td>48633.98</td>
<td>39336.37</td>
</tr>
<tr>
<td>28</td>
<td>3.1600 x 10^{-2}</td>
<td>3.12 x 10^{-2}</td>
<td>45618.48</td>
</tr>
</tbody>
</table>

* Closure Error
1 Kg = 2.2 lb
1 Kgm = 7.217 ft, lb
Example 2.

In the following beam Fig. 7 \( E = 3250 \text{ ksi} \times (2.285 \times 10^5 \text{ Kg/cm}^2) \), \( K = 48 \text{ K/ef} \times (1.77 \text{ Kg/cm}^2) \) and width \( B = 10 \text{ ft} \times (3.048 \text{ m}) \); for comparison this problem has been solved by finite difference [11] and by the imbedding method. In both methods the beam is divided into ten equal sections (Fig. 7). Table 2 shows the shear and bending moment at the nodes. As it can be seen, the moment should be symmetric at the nodes. The symmetry is held in the imbedding method, while in the finite difference method, the symmetry is not held for the moment at symmetric nodes.

![Fig. 7. Forces on the beam for Example 2.](image-url)

![Fig. 8. Beam for Example 3.](image-url)
Sample example 3.

The beam of Fig. (8) with \( E = 4.32 \times 10^4 \) Kips/ft² (2.11x10^5 Kg/cm²), \( I = 5.0625 \) ft⁴ (4.37x10⁶ cm⁴), \( K = 200 \) kip/ft² (3.2 Kg/cm²) and width \( B = 18 \) ft (5.49 m) is considered. The moments and shears are calculated at columns by

(a) Imbedding method with 64 elements, each 1 foot in length.

(b) Method recommended for the solution of combined footings by ACI [3] which is based on the work of Kramerisch and Rogers [4].

Table 2. Comparison of the imbedding and finite difference methods.

<table>
<thead>
<tr>
<th>Node</th>
<th>Imbedding Method</th>
<th>Finite Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( V K_{ip} )</td>
<td>( M ft - K_{ip} )</td>
</tr>
<tr>
<td>1</td>
<td>63.6977</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>185.0697</td>
<td>127.3994</td>
</tr>
<tr>
<td>3</td>
<td>251.7398</td>
<td>56.4934</td>
</tr>
<tr>
<td>4</td>
<td>129.5973</td>
<td>503.8984</td>
</tr>
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<td>5</td>
<td>76.3084</td>
<td>656.5150</td>
</tr>
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<td>566.5150</td>
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<td>503.8984</td>
</tr>
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<td>185.0697</td>
<td>244.7398</td>
</tr>
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<td>129.5973</td>
<td>127.3994</td>
</tr>
<tr>
<td>11</td>
<td>63.6977</td>
<td>8x10⁻⁷</td>
</tr>
</tbody>
</table>

\( \star \) Column Error

1 Kip = 453.59 Kg
1 ft = 138.25 Kg/cm
(c) For comparison, the problem is solved by the Petryi procedure of beam on elastic foundation. The results are shown in Table 3. The results show that the imbedding method is not only easy to use but is also sufficiently accurate.

<table>
<thead>
<tr>
<th>Column</th>
<th>Imbedding meth.</th>
<th>ACI</th>
<th>Herryi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>$M \ k_{p_{G}}$</td>
<td>$M$</td>
<td>$M$</td>
</tr>
<tr>
<td>A</td>
<td>687.00</td>
<td>560</td>
<td>637.74</td>
</tr>
<tr>
<td>B</td>
<td>906.960</td>
<td>879</td>
<td>913.00</td>
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<tr>
<td>C</td>
<td>906.960</td>
<td>879</td>
<td>913.00</td>
</tr>
<tr>
<td>D</td>
<td>687.00</td>
<td>560</td>
<td>637.74</td>
</tr>
</tbody>
</table>

$1 \ k_{p} = 2.21 \text{ kN}$  
$1 \ k_{p} = 7.217 \text{ kN}$

3. DISCUSSION

Considering the theoretical development and sample problems which are solved, it is seen that the imbedding method for solving the problems of beam on Winkler foundation is capable of being solved with a substantial number of nodes and negligible closure errors. In particular, it is seen that the symmetry in the second example is preserved especially in moments by the imbedding method, while the finite difference method has a slight non-symmetry. It is clear that the imbedding method can be used with common desk calculators while the matrix inversion needs a computer. It should be mentioned that the imbedding method can also be applied to the solution of lateral loaded piles and sheet piles. Furthermore, variation of moment of inertia and existence of external moment can easily be incorporated in the calculation procedure by changing and modifying Eq. (4).

4. CONCLUSION

It can be concluded that the imbedding method is capable of solving the problem of beam on Winkler foundation by hand or common desk calcul.
lators and with only small errors of closure. It is further demonstrated that
the imbedding method, in comparison with the methods of finite differences,
that method as recommended by ACI and also by the Hetenyi solution, has
shown very accurate and promising results.

NOMENCLATURE

\begin{align*}
A_j, B_j, C_j, D_j, F_j, F_j' \quad & \text{Imbedding coefficients} \\
B \quad & \text{Width of beam} \\
D \quad & \text{Depth of beam} \\
E \quad & \text{Elastic modulus of concrete} \\
I \quad & \text{Moment of inertia of beam} \\
K \quad & \text{Modulus of subgrade reaction} \\
L \quad & \text{Length of beam} \\
M_j \quad & \text{Moment} \\
M_j \quad & \text{Number of modes} \\
P \quad & \text{Load} \\
P \quad & \text{Pressure} \\
R_j \quad & \text{Reaction} \\
S_{j, j} \quad & \text{Shear force} \\
V_j \quad & \text{Deflection} \\
Y_{j, j} \quad & \text{Second derivative of deflection} \\
\alpha \quad & \text{Parameter in moment equation} \\
\Delta L \quad & \text{Length of each element}
\end{align*}

REFERENCES